The p-adic Generalized Twisted (h,q)-Euler-l-Function and Its Applications

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Abstract : The main purpose of this paper is to construct the p-adic twisted (h,q)-Euler-l-function, which interpolates generalized twisted (h,q)-Euler numbers associated with a primitive Dirichlet character χ. This is a partial answer for the open question which was remained in [13]. An application of this function leads general congruences systems for generalized twisted (h,q)-Euler numbers associated with χ, in particular, Kummer-type congruences for these numbers are obtained.

Keywords : p-adic q-Volkenborn integration, Euler numbers and polynomials, Kummer congruences.

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1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, sets of positive integer, integer, rational, real and complex numbers as usual. Let p be a fixed odd prime number and $x \in \mathbb{Q}$. Then there exists integers m, n and $\nu_p(x)$ such that $x = p^{\nu_p(x)} m/n$ and p does not divide either m or n. Let $|\cdot|_p$ be defined such that $|x|_p = p^{-\nu_p(x)}$ and $|0|_p = 0$. Then $|\cdot|_p$ is a valuation on $\mathbb{Q}$ which satisfies the non-Archimedean property

$$|x+y|_p \leqslant \max\left\{|x|_p, |y|_p\right\}.$$

Completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted by $\mathbb{Q}_p$ and called the field of p-adic rational numbers. But $\mathbb{Q}_p$ itself is not complete with respect to $|\cdot|_p$. $\mathbb{C}_p$ is the completion of the algebraic closure of $\mathbb{Q}_p$ and $\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p : |x|_p \leqslant 1\right\}$ is called the ring of p-adic rational integers (see [14], [17]).

Let d be a fixed positive odd integer and let

$$\begin{aligned}
\mathbb{X} &= \mathbb{X}_d = \varprojlim_N \left(\mathbb{Z}/dp^N\mathbb{Z}\right), \; \mathbb{X}_1 = \mathbb{Z}_p, \\
\mathbb{X}^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} \left(a + dp^N\mathbb{Z}_p\right), \\
a + dp^N\mathbb{Z}_p &= \left\{x \in \mathbb{X} : x \equiv a \left(\mathrm{mod}\, dp^N\right)\right\},
\end{aligned}$$

where $N \in \mathbb{N}$ and $a \in \mathbb{Z}$ with $0 \leqslant a < dp^N$ ([1], [5], [12], [18]).

When talking about q-extensions, q can variously be considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1-q|_p < p^{-1/(p-1)}$ so that for $|x|_p \leqslant 1$, we have $q^x = \exp(x \log q)$ ([1], [2], [5], [6],

[7]). We use the notations

$$[x]_q = \frac{1-q^x}{1-q} \text{ and } [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

We say that f is uniformly differentiable function at a point $a \in \mathbb{X}$, and denote this property by $f \in UD(\mathbb{X})$, if the quotient of the differences

$$F_f = \frac{f(x)-f(y)}{x-y}$$

has a limit $l = f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{X})$, the p-adic invariant q-integral on $\mathbb{X}$ was defined by

$$I_q(f) = \int_{\mathbb{X}} f(t)\, d\mu_q(t) = \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{a=0}^{dp^N-1} f(a)\, q^a$$

(cf. [5], [7]), where for any positive integer N

$$\mu_q\left(a + dp^N \mathbb{Z}_p\right) = \frac{q^a}{[dp^N]_q}$$

(cf. [5], [6], [7]).

The concept of *twisted* has been applied by many authors to certain functions which interpolate certain number sequences. In [15], Koblitz defined twisted Dirichlet L-function which interpolates twisted Bernoulli numbers in the field of complex numbers. In [20], Simsek constructed a q-analogue of the twisted L-function interpolating q-twisted Bernoulli numbers. Kim et.al. [12] derived a p-adic analogue of the twisted L-function by using p-adic invariant integrals. By using the definition of h-extension of p-adic q-L-function which is constructed by Kim [11], Simsek [22, 23] and Jang [4] defined twisted p-adic generalized (h,q)-L-function. In [18], Satoh derived p-adic interpolation function for q-Frobenius-Euler numbers. Simsek [21] gave twisted extensions of q-Frobenius-Euler numbers and their interpolating function q-twisted l-series. In [1], Cenkci et.al. constructed generalized p-adic twisted l-function in p-adic number field. Recently, Kim and Rim [13] defined twisted q-Euler numbers by using p-adic invariant integral on $\mathbb{Z}_p$ in the fermionic sense. In that paper, they raised the following question: *Find a p-adic analogue of the q-twisted l-function which interpolates $E_{n,\xi,q,\chi}^{(h,1)}$, the generalized twisted q-Euler numbers attached to χ [8], [10].* In a forthcoming paper, Rim et.al. [16] answered this question by constructing partial (h,q)-zeta function motivating from a method of Washington [24, 25].

In this paper, we construct p-adic generalized twisted (h,q)-Euler-l-function by employing p-adic invariant measure on p-adic number field. This is the answer of the part of the question posed in [13]. This way of derivation of p-adic generalized twisted (h,q)-Euler-l-function is different from that of [16], and leads an explicit integral representation for this function. As an application of the derived integral representation, we obtain general congruences systems for generalized twisted q-Euler numbers associated with χ, containing Kummer-type congruences.

2. Generalized Twisted q-Euler Numbers

In this section, we give a brief summary of the concepts p-adic q-integrals and Euler numbers and polynomials. Let $UD(\mathbb{X})$ be the set of all uniformly differentiable functions on $\mathbb{X}$. For any $f \in UD(\mathbb{X})$, Kim defined a q-analogue of an integral with respect to a p-adic invariant measure in [5, 7] which was called p-adic q-integral. The p-adic q-integral was defined as follows:

$$I_q(f) = \int_{\mathbb{X}} f(t)\, d\mu_q(t) = \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{a=0}^{dp^N-1} f(a)\, q^a.$$

Note that

$$I_1(f)=\lim_{q\to 1} I_q(f)=\int_{\mathbb{X}} f(t)\,d\mu_1(t)=\lim_{N\to\infty}\frac{1}{dp^N}\sum_{a=0}^{dp^N-1} f(a)$$

is the Volkenborn integral (see [17]).

The Euler zeta function $\zeta_E(s)$ is defined by means of

$$\zeta_E(s)=2\sum_{k=1}^{\infty}\frac{(-1)^k}{k^s}$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$ (cf. [8]). For a Dirichlet character χ with conductor d, $d\in\mathbb{N}$, d is odd, the l-function associated with χ is defined as ([8])

$$l(s,\chi)=2\sum_{k=1}^{\infty}\frac{\chi(k)(-1)^k}{k^s}$$

for $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$. This function can be analytically continued to whole complex plane, except $s=1$ when $\chi=1$; and when $\chi=1$, it reduces to Euler zeta function $\zeta_E(s)$. In [9], (h,q)-extension of Euler zeta function is defined by

$$\zeta_{E,q}^{(h)}(s,x)=[2]_q\sum_{k=0}^{\infty}\frac{(-1)^k q^{hk}}{[k+x]_q^s}$$

with s, $h\in\mathbb{C}$, $\mathrm{Re}(s)>1$ and $x\neq$negative integer or zero. (h,q)-Euler polynomials are defined by the p-adic q-integral as

$$E_{n,q}^{(h)}(x)=\int_{\mathbb{X}} q^{(h-1)t}[t+x]_q^n\,d\mu_{-q}(t),$$

for $h\in\mathbb{Z}$. $E_{n,q}^{(h)}(0)=E_{n,q}^{(h)}$ are called (h,q)-Euler numbers. In [9], it has been shown that for $n\in\mathbb{Z}$, $n\geqslant 0$

$$\zeta_{E,q}^{(h)}(-n,x)=E_{n,q}^{(h)}(x),$$

thus we have

$$E_{n,q}^{(h)}(x)=[2]_q\sum_{k=0}^{\infty}(-1)^k q^{hk}[k+x]_q^n,$$

from which the following entails:

$$E_{n,q}^{(h)}(x)=\frac{[2]_q}{(1-q)^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^j q^{xj}\frac{1}{1+q^{h+j}}.$$

In [8, 9], (h,q)-extension of the l-function associated with χ is defined by

$$l_q^{(h)}(s,\chi)=[2]_q\sum_{k=1}^{\infty}\frac{\chi(k)(-1)^k q^{hk}}{[k]_q^s}$$

for h, $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$. The negative integer values of s are determined explicitly by

$$l_q^{(h)}(-n,\chi)=E_{n,q,\chi}^{(h)},$$

for $n\in\mathbb{Z}$, $n\geqslant 0$ where $E_{n,q,\chi}^{(h)}$ are the generalized (h,q)-Euler numbers associated with χ defined by

$$E_{n,q,\chi}^{(h)}=\int_{\mathbb{X}}\chi(t)\,q^{(h-1)t}[t]_q^n\,d\mu_{-q}(t)\left(=[2]_q\sum_{k=1}^{\infty}\chi(k)(-1)^k q^{hk}[k]_q^n\right).$$

Now assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. From the definition of p-adic invariant q integral on $\mathbb{X}$, Kim [8] defined the integral

$$I_{-1}(f) = \lim_{q \to -1} I_q(f) = \int_{\mathbb{X}} f(t)\, d\mu_{-1}(t) \tag{2.1}$$

for $f \in UD(\mathbb{X})$. Note that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{2.2}$$

where $f_1(t) = f(t+1)$. Repeated application of last formula yields

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{j=0}^{n-1} (-1)^{n-1-j} f(j), \tag{2.3}$$

with $f_n(t) = f(t+n)$.

Let $T_p = \bigcup_{n \geqslant 1} C_{p^n} = \lim_{n \to \infty} \mathbb{Z}/p^n\mathbb{Z}$, where $C_{p^n} = \left\{ w \in \mathbb{X} : w^{p^n} = 1 \right\}$ is the cyclic group of order p^n. For $w \in T_p$, let $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ denote the locally constant function defined by $t \to w^t$.

For $f(t) = \phi_w(t)\, e^{zt}$, we obtain

$$\int_{\mathbb{X}} \phi_w(t)\, e^{zt} d\mu_{-1}(z) = \frac{2}{we^z + 1}$$

using (2.1) and (2.2), and

$$\int_{\mathbb{X}} \chi(t)\, \phi_w(t)\, e^{zt} d\mu_{-1}(t) = 2\sum_{i=1}^{d} \frac{\chi(i)\, \phi_w(i)\, e^{iz}}{w^d e^{dz} + 1}$$

using (2.1) and (2.3) (cf. [8]). As a consequence, the twisted Euler numbers and generalized twisted Euler numbers associated with χ can respectively be defined by

$$\frac{2}{we^z + 1} = \sum_{n=0}^{\infty} E_{n,w} \frac{z^n}{n!}, \text{ and } 2\sum_{i=1}^{d} \frac{\chi(i)\, \phi_w(i)\, e^{iz}}{w^d e^{dz} + 1} = \sum_{n=0}^{\infty} E_{n,w,\chi} \frac{z^n}{n!},$$

from which

$$\int_{\mathbb{X}} t^n \phi_w(t)\, d\mu_{-1}(t) = E_{n,w}, \text{ and } \int_{\mathbb{X}} \chi(t)\, t^n \phi_w(t)\, d\mu_{-1}(t) = E_{n,w,\chi}$$

follow.

Twisted extension of (h,q)-Euler zeta function is defined by

$$\zeta_{E,q,w}^{(h)}(s,x) = [2]_q \sum_{k=0}^{\infty} \frac{(-1)^k w^k q^{hk}}{[k+x]_q^s}$$

with h, $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$ and $x \neq$negative integer or zero. For $n \in \mathbb{Z}$, $n \geqslant 0$ and $h \in \mathbb{Z}$, this function gives

$$\zeta_{E,q,w}^{(h)}(-n,x) = E_{n,q,w}^{(h)}(x),$$

where $E_{n,q,w}^{(h)}(x)$ are the twisted q-Euler polynomials defined as

$$E_{n,q,w}^{(h)}(x) = \int_{\mathbb{X}} q^{(h-1)t} \phi_w(t)\, [x+t]_q^n \, d\mu_{-q}(t) \left(= [2]_q \sum_{k=0}^{\infty} (-1)^k w^k q^{hk} [k+x]_q^n \right)$$

by using p-adic invariant q-integral on $\mathbb{X}$ in the fermionic sense (cf. [13], [16]). The following expressions for twisted (h,q)-Euler polynomials can be verified from the defining equalities:

$$E_{n,q,w}^{(h)}(x) = \frac{[2]_q}{(1-q)^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^j q^{xj}\frac{1}{1+wq^{h+j}}, \tag{2.4}$$

$$E_{n,q,w}^{(h)}(x) = \frac{[2]_q}{[2]_{q^d}}[d]_q^n\sum_{a=0}^{d-1}q^{ha}w^a(-1)^a E_{n,q^d,w^d}^{(h)}\left(\frac{x+a}{d}\right), \tag{2.5}$$

where n, $d\in\mathbb{N}$ with d is odd. From (2.4), the twisted (h,q)-Euler polynomials can be determined explicitly. A few of them are

$$\begin{aligned}
E_{0,q,w}^{(h)}(x) &= \frac{1+q}{1+wq^h},\\
E_{1,q,w}^{(h)}(x) &= \frac{1+q}{1-q}\left(\frac{1}{1+wq^h}-\frac{q^x}{1+wq^{h+1}}\right),\\
E_{2,q,w}^{(h)}(x) &= \frac{1+q}{(1-q)^2}\left(\frac{1}{1+wq^h}-\frac{2q^x}{1+wq^{h+1}}+\frac{q^{2x}}{1+wq^{h+2}}\right).
\end{aligned}$$

For $x=0$, $E_{n,q,w}^{(h)}(0)=E_{n,q,w}^{(h)}$ are called twisted (h,q)-Euler numbers. Thus we can write

$$E_{n,q,w}^{(h)}(x)=\sum_{j=0}^{n}\binom{n}{j}q^{xj}[x]_q^{n-j}E_{j,q,w}^{(h)}.$$

Let χ be a Dirichlet character of conductor d with $d\in\mathbb{N}$ and d is odd. Then the generalized twisted (h,q)-Euler numbers associated with χ are defined as

$$E_{n,q,w,\chi}^{(h)}=\int_{\mathbb{X}}\chi(t)\,q^{(h-1)t}\phi_w(t)\,[t]_q^n\,d\mu_{-q}(t).$$

These numbers arise at the negative integer values of the twisted (h,q)-Euler-l-function which is defined by

$$l_{q,w}^{(h)}(s,\chi)=[2]_q\sum_{k=1}^{\infty}\frac{\chi(k)(-1)^k w^k q^{hk}}{[k]_q^s}$$

with h, $s\in\mathbb{C}$, $\mathrm{Re}(s)>1$. Indeed, for $n\in\mathbb{Z}$, $n\geqslant 0$ and $h\in\mathbb{Z}$, we have

$$l_{q,w}^{(h)}(-n,\chi)=E_{n,q,w,\chi}^{(h)}$$

(cf. [13], [16]).

We conclude this section by stating the distribution property for generalized twisted (h,q)-Euler numbers associated with χ, which will take a major role in constructing a measure in the next section.

For n, $d\in\mathbb{N}$ with d is odd, we have

$$E_{n,q,w,\chi}^{(h)}=\frac{[2]_q}{[2]_{q^d}}[d]_q^n\sum_{a=1}^{d}q^{ha}w^a\chi(a)(-1)^a E_{n,q^d,w^d}^{(h)}\left(\frac{a}{d}\right).$$

3. p-adic Twisted (h,q)-l-Functions

In this section we first focus on defining a p-adic invariant measure, which is apparently an important tool to construct p-adic twisted (h,q)-Euler-l-function in the sense of p-adic invariant

q-integral. We afterwards give the definition of p-adic twisted (h,q)-Euler-l-function, together with Witt's type formulas for twisted and generalized twisted (h,q)-Euler numbers.

Throughout, we assume that ξ is the rth root of unity with $(r,pd)=1$, where p is an odd prime and d is an odd natural number. If $(r,pd)=1$, it has been known that $|1-\xi|_p \geqslant 1$ (see [15], [19]) and ξ lies in the cyclic group $C_{p^n}=\left\{w : w^{p^n}=1\right\}$. The following theorem plays a crucial role in constructing p-adic generalized twisted (h,q)-Euler-l-function on $\mathbb{X}$.

Theorem 3.1 *Let $q\in\mathbb{C}_p$ with $|1-q|_p<p^{-1/(p-1)}$ and ξ is the rth root of unity with $|1-\xi|_p\geqslant 1$. For $N\in\mathbb{Z}$, $n\in\mathbb{Z}$, $n\geqslant 0$, let $\mu_{n,\xi,q}^{(h)}$ be defined as*

$$\mu_{n,\xi,q}^{(h)}\left(a+dp^N\mathbb{Z}_p\right)=\left[dp^N\right]_q^n\frac{[2]_q}{[2]_{q^{dp^N}}}(-1)^a\xi^a q^{ha}E_{n,q^{dp^N},\xi^{dp^N}}^{(h)}\left(\frac{a}{dp^N}\right).$$

Then $\mu_{n,\xi,q}^{(h)}$ extends uniquely to a measure on $\mathbb{X}$.

Proof. In order to show that $\mu_{n,\xi,q}^{(h)}$ is a measure on $\mathbb{X}$, we need to show that it is a distribution and is bounded on $\mathbb{X}$.

To show it is a distribution on $\mathbb{X}$, we check the equality

$$\sum_{i=0}^{p-1}\mu_{n,\xi,q}^{(h)}\left(a+idp^N+dp^{N+1}\mathbb{Z}_p\right)=\mu_{n,\xi,q}^{(h)}\left(a+dp^N\mathbb{Z}_p\right).$$

Beginning the calculation from right hand side yields

$$\begin{aligned}
&\sum_{i=0}^{p-1}\mu_{n,\xi,q}^{(h)}\left(a+idp^N+dp^{N+1}\mathbb{Z}_p\right)\\
&=\sum_{i=0}^{p-1}\left[dp^{N+1}\right]_q^n\frac{[2]_q}{[2]_{q^{dp^{N+1}}}}(-1)^{a+idp^N}\xi^{a+idp^N}q^{h\left(a+idp^N\right)}E_{n,q^{dp^{N+1}},\xi^{dp^{N+1}}}^{(h)}\left(\frac{a+idp^N}{dp^{N+1}}\right)\\
&=\left[dp^N\right]_q^n\frac{[2]_q}{[2]_{q^{dp^N}}}(-1)^a\xi^a q^{ha}[p]_{q^{dp^N}}^n\frac{[2]_{q^{dp^N}}}{[2]_{q^{dp^{N+1}}}}\sum_{i=0}^{p-1}(-1)^i\left(\xi^{dp^N}\right)^i\left(q^{dp^N}\right)^{hi}\\
&\quad\times\quad E_{n,\left(q^{dp^N}\right)^p,\left(\xi^{dp^N}\right)^p}^{(h)}\left(\frac{\frac{a}{dp^N}+i}{p}\right)\\
&=\left[dp^N\right]_q^n\frac{[2]_q}{[2]_{q^{dp^N}}}(-1)^a\xi^a q^{ha}E_{n,q^{dp^N},\xi^{dp^N}}^{(h)}\left(\frac{a}{dp^N}\right)=\mu_{n,\xi,q}^{(h)}\left(a+dp^N\mathbb{Z}_p\right),
\end{aligned}$$

where we have used (2.5).

To present boundedness, we use equation (2.4) to expand the polynomial $E_{n,q^{dp^N},\xi^{dp^N}}^{(h)}\left(\frac{a}{dp^N}\right)$, so that

$$\mu_{n,\xi,q}^{(h)}\left(a+dp^N\mathbb{Z}_p\right)=\frac{[2]_q}{(1-q)^n}(-1)^a\xi^a q^{ha}\sum_{j=0}^{n}\binom{n}{j}(-1)^j q^{ja}\frac{1}{1+\xi^{dp^N}q^{hdp^N+jdp^N}}.$$

Now, since d is an odd natural number and p is an odd prime, we have $\left|1-\left(-\xi^{dp^N}q^{hdp^N+jdp^N}\right)\right|_p\geqslant 1$, so by induction on j, we obtain

$$\left|\mu_{n,\xi,q}^{(h)}\left(a+dp^N\mathbb{Z}_p\right)\right|_p\leqslant M$$

for a constant M. This is what we require, so the proof is completed. ■

Let χ be a Dirichlet character with conductor d. Then we can express the generalized twisted (h,q)-Euler numbers associated with χ as an integral over $\mathbb{X}$, by using the measure $\mu_{n,\xi,q}^{(h)}$.

Lemma 3.2 *For $n \in \mathbb{Z}$, $n \geqslant 0$, we have*

$$\int_{\mathbb{X}} \chi(t)\, d\mu_{n,\xi,q}^{(h)}(t) = E_{n,q,\xi,\chi}^{(h)}.$$

Proof. From the definition of p-adic invariant integral, we have

$$\int_{\mathbb{X}} \chi(t)\, d\mu_{n,\xi,q}^{(h)}(t) = \lim_{N\to\infty} \sum_{c=0}^{dp^N-1} \chi(c) \left[dp^N\right]_q^n \frac{[2]_q}{[2]_{q^{dp^N}}} (-1)^c \xi^c q^{hc} E_{n,q^{dp^N},\xi^{dp^N}}^{(h)} \left(\frac{c}{dp^N}\right).$$

Writing $c = a + dm$ with $a = 0, 1, \ldots, d-1$ and $m = 0, 1, 2, \ldots$, we get

$$\begin{aligned}
\int_{\mathbb{X}} \chi(t)\, d\mu_{n,\xi,q}^{(h)}(t) &= [d]_q^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} \chi(a) (-1)^a \xi^a q^{ha} \\
&\times \lim_{N\to\infty} \left[p^N\right]_{q^d}^n \frac{[2]_{q^d}}{[2]_{(q^d)^{p^N}}} \sum_{m=0}^{p^N-1} (-1)^m \left(\xi^d\right)^m \left(q^d\right)^{hm} E_{n,(q^d)^{p^N},(\xi^d)^{p^N}}^{(h)} \left(\frac{\frac{a}{d}+m}{p^N}\right) \\
&= [d]_q^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} \chi(a) (-1)^a \xi^a q^{ha} E_{n,q^d,\xi^d}^{(h)} \left(\frac{a}{d}\right).
\end{aligned}$$

Assuming $\chi(0) = 0$ and by the fact that $\chi(d) = 0$, last expression equals $E_{n,\xi,q,\chi}^{(h)}$, and the proof is completed. ■

Since it is impossible to have a non-zero translation-invariant measure on $\mathbb{X}$, $\mu_{n,\xi,q}^{(h)}$ is not invariant under translation, but satisfies the following:

Lemma 3.3 *For a compact-open subset U of $\mathbb{X}$, we have*

$$\mu_{n,\xi,q}^{(h)}(pU) = [p]_q^n \frac{[2]_q}{[2]_{q^p}} \mu_{n,\xi^p,q^p}^{(h)}(U).$$

Proof. Let $U = a + dp^N\mathbb{Z}_p$ be the compact-open subset of $\mathbb{X}$. Then

$$\begin{aligned}
\mu_{n,\xi,q}^{(h)}(pU) &= \mu_{n,\xi,q}^{(h)}\left(pa + dp^{N+1}\mathbb{Z}_p\right) \\
&= \left[dp^{N+1}\right]_q^n \frac{[2]_q}{[2]_{q^{dp^{N+1}}}} (-1)^{pa} \xi^{pa} q^{hpa} E_{n,q^{dp^{N+1}},\xi^{dp^{N+1}}}^{(h)} \left(\frac{pa}{dp^{N+1}}\right) \\
&= \left[p^N\right]_q^n \frac{[2]_q}{[2]_{q^p}} \left[dp^N\right]_{q^p}^n \frac{[2]_{q^p}}{[2]_{(q^p)^{dp^N}}} (-1)^a (\xi^p)^a (q^p)^{ha} E_{n,(q^p)^{dp^N},(\xi^p)^{dp^N}}^{(h)} \left(\frac{a}{dp^N}\right) \\
&= \left[p^N\right]_q^n \frac{[2]_q}{[2]_{q^p}} \mu_{n,\xi^p,q^p}^{(h)}\left(a + dp^N\mathbb{Z}_p\right) = [p]_q^n \frac{[2]_q}{[2]_{q^p}} \mu_{n,\xi^p,q^p}^{(h)}(U),
\end{aligned}$$

which is the desired result. ■

Next, we give a relation between $\mu_{n,\xi,q}^{(h)}$ and μ_{-q}.

Lemma 3.4 *For any $n \in \mathbb{Z}$, $n \geqslant 0$, we have*

$$d\mu_{n,\xi,q}^{(h)}(t) = q^{(h-1)t}\xi^t [t]_q^n \, d\mu_{-q}(t).$$

Proof. From the definition of $\mu_{n,\xi,q}^{(h)}$ and expansion of twisted (h,q)-Euler polynomials, we have

$$\mu_{n,\xi,q}^{(h)}\left(a + dp^N\mathbb{Z}_p\right) = \frac{[2]_q}{(1-q)^n}(-1)^a \xi^a q^{ha} \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{ja} \frac{1}{1+\xi^{dp^N} q^{hdp^N + jdp^N}}.$$

By the same method presented in [7], we obtain

$$\begin{aligned}
\lim_{N\to\infty} \mu_{n,\xi,q}^{(h)}\left(a + dp^N\mathbb{Z}_p\right) &= \frac{1}{2}\frac{[2]_q}{(1-q)^n}(-1)^a \xi^a q^{ha} \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{ja} \\
&= \frac{1+q}{2}\xi^a q^{(h-1)a} [a]_q^n (-1)^a q^a = q^{(h-1)a}\xi^a [a]_q^n \lim_{N\to\infty} \frac{(-1)^a q^a}{\frac{1-\left(-q^{dp^N}\right)}{1-(-q)}} \\
&= q^{(h-1)a}\xi^a [a]_q^n \lim_{N\to\infty} \mu_{-q}\left(a + dp^N\mathbb{Z}_p\right).
\end{aligned}$$

We thus have

$$d\mu_{n,\xi,q}^{(h)}(t) = q^{(h-1)t}\xi^t [t]_q^n \, d\mu_{-q}(t),$$

the desired result. ■

Let ω denote the Teichmüller character mod p. For an arbitrary character χ and $n \in \mathbb{Z}$, let $\chi_n = \chi\omega^{-n}$ in the sense of product of characters. For $t \in \mathbb{X}^* = \mathbb{X} - p\mathbb{X}$, we set $\langle t \rangle_q = [t]_q / \omega(t)$. Since $\left| \langle t \rangle_q - 1 \right|_p < p^{-1/(p-1)}$, $\langle t \rangle_q^s$ is defined by $\exp\left(s \log_p \langle t \rangle_q\right)$ for $|s|_p \leqslant 1$, where $\log_p$ is the Iwasawa p-adic logarithm function ([3]). For $|1-q|_p < p^{-1/(p-1)}$, we have $\langle t \rangle_q^{p^N} \equiv 1 \left(\mathrm{mod} p^N\right)$.

We now define p-adic generalized twisted (h,q)-Euler-l-function.

Definition 3.5 *For $s \in \mathbb{Z}_p$,*

$$l_{p,q,\xi}^{(h)}(s,\chi) = \int_{\mathbb{X}^*} \langle t \rangle_q^{-s} q^{(h-1)t}\xi^t d\mu_{-q}(t).$$

The values of this function at non-positive integers are given by the following theorem:

Theorem 3.6 *For any $n \in \mathbb{Z}$, $n \geqslant 0$,*

$$l_{p,q,\xi}^{(h)}(-n,\chi) = E_{n,q,\xi,\chi_n}^{(h)} - \chi_n(p)[p]_q^n \frac{[2]_q}{[2]_{q^p}} E_{n,q^p,\xi^p,\chi_n}^{(h)}.$$

Proof.

$$\begin{aligned}
l_{p,q,\xi}^{(h)}(-n,\chi) &= \int_{\mathbb{X}^*} \langle t \rangle_q^n q^{(h-1)t}\xi^t d\mu_{-q}(t) = \int_{\mathbb{X}^*} \chi_n(t) [t]_q^n q^{(h-1)t}\xi^t d\mu_{-q}(t) \\
&= \int_{\mathbb{X}^*} \chi_n(t) \, d\mu_{n,\xi,q}^{(h)}(t) = \int_{\mathbb{X}} \chi_n(t) \, d\mu_{n,\xi,q}^{(h)}(t) - \int_{p\mathbb{X}} \chi_n(t) \, d\mu_{n,\xi,q}^{(h)}(t) \\
&= E_{n,q,\xi,\chi_n}^{(h)} - \chi_n(p)[p]_q^n \frac{[2]_q}{[2]_{q^p}} E_{n,q^p,\xi^p,\chi_n}^{(h)},
\end{aligned}$$

where Lemma 3.2, Lemma 3.3 and Lemma 3.4 are used. ■

This theorem will be mainly used in the next section, where certain applications of p-adic generalized twisted (h,q)-Euler-l-function are given.

4. Kummer Congruences for Generalized Twisted (h, q)-Euler Numbers

This section is devoted to an application of the p-adic generalized twisted (h, q)-Euler-l-function to an important number theoretic concept, congruences systems. In particular, we derive Kummer-type congruences for generalized twisted (h, q)-Euler numbers by using p-adic integral representation of p-adic generalized twisted (h, q)-Euler-l-function and Theorem 3.6.

In the sequel, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Then $q \equiv 1 \,(\mathrm{mod}\mathbb{Z}_p)$. For $t \in \mathbb{X}^*$, we have $[t]_q \equiv t \,(\mathrm{mod}\mathbb{Z}_p)$, thus $\langle t \rangle_q \equiv 1 \,(\mathrm{mod}p\mathbb{Z}_p)$. For a positive integer c, the forward difference operator Δ_c acts on a sequence $\{a_m\}$ by $\Delta_c a_m = a_{m+c} - a_m$. The powers Δ_c^k of Δ_c are defined by Δ_c^0 =identity and for any positive integer k, $\Delta_c^k = \Delta_c \circ \Delta_c^{k-1}$. Thus

$$\Delta_c^k a_m = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \, a_{m+jc}.$$

For simplicity in the notation, we write

$$\varepsilon_{n,q,\xi,\chi_n}^{(h)} = E_{n,q,\xi,\chi_n}^{(h)} - \chi_n(p) \, [p]_q^n \, \frac{[2]_q}{[2]_{q^p}} E_{n,q^p,\xi^p,\chi_n}^{(h)}.$$

Theorem 4.1 *For $n \in \mathbb{Z}$, $n \geqslant 0$ and $c \equiv 0 \,(mod\,(p-1))$, we have*

$$\Delta_c^k \varepsilon_{n,q,\xi,\chi_n}^{(h)} \equiv 0 \left(mod p^k \mathbb{Z}_p\right).$$

Proof. Since Δ_c^k is a linear operator, by Theorem 3.6 we have

$$\begin{aligned} \Delta_c^k \varepsilon_{n,q,\xi,\chi_n}^{(h)} &= \Delta_c^k l_{p,q,\xi}^{(h)}(-n,\chi) = \Delta_c^k \int_{\mathbb{X}^*} \langle t \rangle_q^n \, q^{(h-1)t} \xi^t d\mu_{-q}(t) \\ &= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \int_{\mathbb{X}^*} \langle t \rangle_q^{n+jc} \, q^{(h-1)t} \xi^t d\mu_{-q}(t) \\ &= \int_{\mathbb{X}^*} \langle t \rangle_q^n \, q^{(h-1)t} \xi^t \left(\langle t \rangle_q^c - 1\right)^k d\mu_{-q}(t). \end{aligned}$$

Now, $\langle t \rangle_q \equiv 1 \,(\mathrm{mod}p\mathbb{Z}_p)$, which implies that $\langle t \rangle_q^c \equiv 1 \,(\mathrm{mod}p\mathbb{Z}_p)$ since $c \equiv 0 \,(\mathrm{mod}\,(p-1))$, and thus

$$\left(\langle t \rangle_q^c - 1\right)^k \equiv 0 \left(\mathrm{mod}p^k \mathbb{Z}_p\right).$$

Therefore

$$\Delta_c^k l_{p,q,\xi}^{(h)}(-n,\chi) \equiv 0 \left(\mathrm{mod}p^k \mathbb{Z}_p\right),$$

from which the result follows. ■

Theorem 4.2 *Let n and n' be positive integers such that $n \equiv n' \,(mod\,(p-1))$. Then, we have*

$$\varepsilon_{n,q,\xi,\chi_n}^{(h)} \equiv \varepsilon_{n',q,\xi,\chi_{n'}}^{(h)} \,(mod p\mathbb{Z}_p).$$

Proof. Without loss of generality, let $n \geqslant n'$. Then

$$l_{p,q,\xi}^{(h)}(-n,\chi) - l_{p,q,\xi}^{(h)}(-n',\chi) = \int_{\mathbb{X}^*} \langle t \rangle_q^n \, q^{(h-1)t} \xi^t \left(\langle t \rangle_q^{n-n'} - 1\right) d\mu_{-q}(t).$$

Since $n - n' \equiv 0 \,(\mathrm{mod}\,(p-1))$, we have $\langle t \rangle_q^{n-n'} - 1 \equiv 0 \,(\mathrm{mod}p\mathbb{Z}_p)$, which entails the result. ■

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